UNIVERSIDAD CARLOS III DE MADRID

working papers

Working Paper 07-35
Economic Series 20
April 26, 2007

Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34-91) 6249875

ON THE GENERIC FINITENESS OF OUTCOME DISTRIBUTIONS FOR BIMATRIX GAME FORMS

NIKOLAI S. KUKUSHKIN, CRISTIAN M. LITAN, AND FRANCISCO MARHUENDA

Abstract

We provide an example of an outcome game form with two players for which there is in an open set of utilities for both players such that, in each of the associated games, the set of Nash equilibria induce a continuum of outcome distributions. The case for three or more players has been settled by Govindan and McLennan [3].

Keywords and phrases: Generic finiteness, Nash equilibrium.

This problem was communicated to us by Francesco De Sinopoli. We thank Andreu Mas-Colell for providing us with the manuscript [7]. The authors aknowledge financial support from the project No. SEJ2004-00968/ECON from the Spanish Ministry of Education and Science. C. Litan acknowledges financial support from "Plan de Formación de Personal Investigador" of the Comunidad de Madrid.

1. Introduction

If one perturbs independently the values of the utility functions in a finite normal form game, then generically there is a finite number of equilibria (see Harsanyi [4]).

However, when the normal form is derived from an extensive form, Harsanyi's result has no immediate implications, because many strategies lead to the same final node. Even if the payoffs of the final nodes can be perturbed independently, the finiteness of the number of equilibria is not necessarily a generic property. There can be constructed many examples of extensive forms where it is not. On such grounds, Kreps and Wilson [5] criticized Harsanyi's result as not being useful for extensive forms.

As noted by Govindan and McLennan [3], "Kreps and Wilson [5] proved a theorem that responds to this critique." Govindan and McLennan [2] made a direct proof of the same result. This result states that if the payoffs of the final nodes can be perturbed independently, the finiteness of the distributions on the terminal nodes (paths) induced by sequential equilibria is a generic property. And this remains true if one considers paths induced by Nash equilibria.

However, as noted by Mas-Colell [7], "the Kreps and Wilson [5] criticism to the normal form result can be reiterated." Govindan and McLennan [3] pointed out in the same direction: "Many games arising in economic models, and in other contexts, have sets of terminal nodes that are naturally regarded as a priori equivalent for reasons arising out of the nature of the phenomenon being modeled."

Therefore, the case in which the payoffs of the final nodes are tied together by convex linear constraints (see Govindan and McLennan [3], [1] and Mas-Colell [7]) is of high conceptual interest. As further noted by Govindan and McLennan [3], in this case "...the 'relevant' space of terminal node utilities is a linear subspace of the space of all terminal node utilities, and generic finiteness in the subspace does not follow from Kreps and Wilson's theorem."

This note continues the above line of research in that it addresses the following question: is it the case that, in the relevant space of terminal node utilities (i.e. when the payoffs of the final nodes are tied together by convex linear constraints), generically there are only finitely many outcome distributions (i.e. distributions on the relevant set of terminal nodes) induced by the Nash equilibria?

When the number of players is strictly greater than two, Govindan and McLennan [3] provide an example of a game form that shows that, in open set of utilities, the Nash equilibria of the associated game induce a continuum of outcome distributions. In that work, they also prove that in some special cases (e.g. with two outcomes and any number of players) the answer to the above question is positive.

However, for games with two players, the above issue has been only partially addressed and the general question for this case has remained unanswered until now. Mas–Colell [7] shows that for two person games the equilibrium payoffs are generically finite. Govindan and McLennan [1] prove that for every bimatrix game form, the equilibrium distributions on outcomes are generically finite in the space of zero sum and common interest utility games.

In this note, we show that the answer to the previous question continues to be negative for two player games. We provide an example of a game form and an open set of utilities, such that in each of the associated games there is a continuum of outcome distributions induced by the Nash equilibria.

2. The conjecture

Let Ω be a finite outcome space and let S_1, S_2 denote the finite strategy spaces of each of the two players. An outcome game form is a mapping $\theta : S_1 \times S_2 \to \Omega$. The utilities of the players are defined by the functions $u_i : \Omega \to \mathbb{R}$, $i = 1, 2$.

For each $i = 1, 2$, let $\Delta_i = \{\mu \in \mathbb{R}^{S_i}_{++} : \sum_{x \in S_i} \mu(x) = 1\}$. A pair of strategy vectors $p_i \in \Delta_i$ of the players induces a probability distribution in Ω.

An outcome game form θ and the utilities of the players u_i, $i = 1, 2$ determine a bimatrix game. We denote by $A(u_i)$ the associated matrices of utilities of the players in this game. That is, the entry a_{jl} of $A(u_i)$ is $u_i(\theta(s_j, s_l))$.

A completely mixed Nash equilibrium in this game consists of two strategy vectors $p_1 \in \Delta_1$ and $p_2 \in \Delta_2$ such that

$$\begin{aligned} A(u_2)\, p_1 &= \beta\, e \quad \text{for some } \beta \in \mathbb{R}. \\ p_2\, A(u_1) &= \alpha\, e \quad \text{for some } \alpha \in \mathbb{R}. \end{aligned}$$

where e denotes the vector (in the appropriate Euclidean space) with all of its entries equal to 1.

Conjecture 1. For every bimatrix game form θ, there is a generic set of utilities in $\mathbb{R}^\Omega \times \mathbb{R}^\Omega$ for which the set of distributions induced on outcomes by the **completely mixed Nash equilibria** is finite.

For the case of two person games, this conjecture corresponds to Conjecture T in Govindan and McLennan [3]. They disprove the conjecture for games with more than two players.

3. The example

We now give an example that shows that Conjecture 1 does not hold. Let there be four outcomes, denoted by $\Omega = \{a, b, c, d\}$. And consider the outcome matrix

$$A(a, b, c, d) = \begin{pmatrix} c & a & b & b \\ d & a & a & b \\ c & d & b & c \end{pmatrix}$$

Let us use the notation $a_i = u_i(a)$, $b_i = u_i(b)$, $c_i = u_i(c)$, $d_i = u_i(d)$, and $u^i = (a_i, b_i, c_i, d_i) \in \mathbb{R}^4$ for the utilities of agent $i = 1, 2$. We denote

$G = \{(u^1, u^2) \in \mathbb{R}^8 \mid d_1, b_1 < a_1, c_1 \ \& \ d_2 < b_2 < a_2, c_2\}$, an open subset in the space of utilities. For each $(u^1, u^2) \in G$ and $t \in \mathbb{R}$, we define

$$p_1(u^2) = \frac{1}{a_2 - b_2 + c_2 - d_2} (b_2 - d_2, c_2 - b_2, a_2 - b_2) \in \mathbb{R}^3$$

and

$$\begin{aligned} p_2(u^1; t) = \Big(& \frac{a_1 - b_1}{a_1 - b_1 + c_1 - d_1} - \frac{(a_1 - b_1)t}{a_1 - d_1}, \ \frac{(c_1 - d_1)t}{a_1 - d_1}, \\ & \qquad \frac{c_1 - d_1}{a_1 - b_1 + c_1 - d_1} - \frac{(c_1 - d_1)t}{a_1 - d_1}, \ t \Big) \in \mathbb{R}^4. \end{aligned}$$

One checks immediately that the pair $\langle p_1(u^2), p_2(u^1; t) \rangle$ is a completely mixed Nash equilibrium provided t is positive and small enough. The probability of outcome a induced by the equilibrium is computed easily as

$$\begin{aligned} p_a = & \frac{(b_2 - c_2)(c_1 - d_1)}{(a_1 - b_1 + c_1 - d_1)(-a_2 + b_2 - c_2 + d_2)} \\ & + \frac{b_2(d_1 - c_1) + b_1(c_2 - d_2) + c_1 d_2 - c_2 d_1}{(a_1 - d_1)(-a_2 + b_2 - c_2 + d_2)} t \end{aligned}$$

Therefore, there is a continuum of equilibrium probability distributions (i.e. for t positive and small enough) on the set of outcomes as long as $(u^1, u^2) \in G$ and $b_2(d_1 - c_1) + b_1(c_2 - d_2) + c_1 d_2 - c_2 d_1 \neq 0$; such values of u^1 and u^2 form an open subset in the space of utilities. Remark that when $u^1 = \pm u^2$, the coefficient of t in the above expression for p_a (as well as for the other outcomes) disappears and there is a unique probability distribution on outcomes, as proved in Govindan and McLennan [1] and Litan [6].

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Russian Academy of Sciences, Dorodnicyn Computing Center, Moscow
E-mail address: ququ@bluebottle.com

University Carlos III de Madrid, Department of Economics, Madrid.
E-mail address: clitan@eco.uc3m.es

University Carlos III de Madrid, Department of Economics, Madrid.
E-mail address: marhuend@eco.uc3m.es